The dialectic between knowledge, knowing, and concept in the theory of objectification

La dialectique entre savoir, connaissance et concept dans la théorie de l'objectivation

Luis Radford
Laurentian University, Ontario, Canada

The theory of objectification (TO) is a theory of learning that seeks to foster conceptually rich, and critical, inclusive, and democratic pedagogical practices. The conception of these practices is based on a new understanding of learning as a cultural-historical process. In turn, the theoretical formulation of learning is anchored in a conception of knowledge that departs from the accounts of rationalists and (new and old) empiricists. The purpose of this article is to offer an overview of knowledge and learning as conceived in the TO. This overview is, of necessity, philosophical, as it addresses a problem that has often been overlooked in educational research: the *ontological* problem of the nature of knowledge—such as mathematical and scientific knowledge. The philosophical overview presented here is based on a specific philosophy that inspires the theory of objectification: dialectical materialism. Drawing on this philosophy, I theorize learning as a social, embodied, affective, semiotic, and material process where individuals encounter knowledge. In this encounter knowledge manifests itself in sensible practical and material activity through what it is called here knowing and concept. As argued in this article, knowledge, knowing, and concept are three modes of existence of a same entity that is invoked in the movement of learning.

Keywords: learning, dialectical materialism, cultural-historical approach, Ilyenkov, Hegel, Vygotsky

Introduction

Educational reflections on the epistemology of mathematics have contributed to clarify several aspects of the conceptual density of mathematical knowledge (Barbin, 2009 ; Bkouche, 1997; Fried, 2009). However, educational reflections have shown limited interest in tackling the *ontological* question of the nature of knowledge in general, and mathematical knowledge in particular (notable exceptions are Otte [2003] and Freitas et al. [2017]). This limited interest could be attributed to the philosophies that have influenced, often implicitly, reflections on the teaching and learning of mathematics. In fact, the predominant philosophies in our research field have been, usually implicitly, empiricism and rationalism. While the former tends to regard knowledge as a subjective entity, as American constructivism in all its variants does (see, for example, von Glasersfeld, 1995), the latter considers knowledge as teleologically driven by a universal inner reason (Husserl, 1972). Ultimately, the result of both approaches has been the same: knowledge is conceived as something that transcends culture.

And this would not be a problem if it were not for the fact that, precisely, sociocultural currents advocate a different understanding of knowledge, arguing that its intrinsic nature is to be *culturally situated*. The extensive range of results contributed by ethnomathematics in recent years supports this idea (Parra, 2018; Rosa et al., 2017). However, despite the advances made in ethnomathematics, there still remains the theoretical

problem of providing a clear conception of what knowledge means in its ontological constitution.

In this paper I offer a philosophical exposition of the idea of knowledge as it has been developed in the theory of objectification from its dialectical materialist basis. The question that can be asked here is, why dialectical materialism? Why not another philosophy? The answer is the following. Dialectical materialism has the advantage of offering a historical and monistic view of the world, of being and knowing. In this view, the individual is not considered as someone whose most vital resources have little to do with their social, cultural, and historical context—the individual *simpliciter* of new and old empiricisms. On the contrary, dialectical materialism offers a view in which knowing and being determine each other in a dialectic of negativities and positivities. It is in this sense that contemporary dialectical materialism (Balibar, 2002; Fischbach, 2014; Levant, 2016) provides us with ideas for thinking of learning as a movement of subjectivities in the making within a dialectics in which knowledge is cultural through and through. Knowledge is what it is through the mediation of being and vice versa. Indeed, dialectical materialism offers us a conception of knowledge that is neither a subjective construction, nor an ahistorical Platonic form, nor a purified abstraction from the concrete world. As the Hegelian philosopher Evald Ilyenkov states, as soon as one attempts to conceive of knowledge as purified of its concrete history and "purified of all the traces of palpable corporeality, it turns out that this attempt is fundamentally doomed to failure, that after such a purification there will be nothing but transparent emptiness, an indefinable vacuum" (2012, p. 177).

Knowledge

Knowledge as a cultural disposition

In the theory of objectification, knowledge is conceived as a complex of *historical-cultural ideas* that function as a *general disposition* to act, understand, interpret, talk, and transform the world.

To understand this idea of knowledge as a cultural disposition, let us imagine two scenarios: the first is a rural community that has developed ideas about time, space, numbers, seed sowing, fishing, etc. The second is a community based on capitalist forms of market production, as in a contemporary European or North American country. Now let us imagine two babies born at the same time in each of these communities. Both babies will encounter a different complex of cultural ideas. For example, a baby born in the Atlantic Canadian Mi'kmaw community described by Lunney Borden (2013) will grow up counting things differently than a New York baby. While the latter will grow up understanding that smell, colour, size, and the kinds of things they count do not matter, the former will not, for "in Mi'kmaq *what* one counts determines *how* one counts" (p. 6; emphasis in the original). As they grow up, each of these babies will be *profoundly influenced* by the corresponding cultural knowledges in their ways of seeing the world and seeing themselves.

We can better understand the idea of knowledge as a cultural disposition if we see it under the category of what Georg Wilhelm Hegel calls *potentiality*. In Hegel's philosophy, potentiality expresses the theoretical view that "nothing is *immediate*" (Hegel, 1969, p. 107; emphasis in original). Every action, every thought, always starts from something else and is, therefore, "the result of *mediation*" (p. 107; emphasis in the original). Thus, when Descartes pronounces his famous "*cogito, ergo sum*," I think, therefore I am, the French philosopher assumes that the connection between thinking and existing resides in the simple intuition of consciousness that he takes as absolutely first. However, instead of being absolutely first, this connection is the *effect* of cultural-historical mediations that make *possible* the new form of subjectivity that appears in the early modern European period and that Descartes himself enunciates. This does not mean that there is nothing new in Descartes's statement. What it does mean is that this newness does not fall from the sky but is shaped by a series of cultural-historical dispositions. The same can be said of Mozart's genius. His genius is not simply the product of an innate predisposition but of an ontogenetic movement in which a special talent is intertwined with the canons of behavior and musical taste of the ruling class of his time. Mozart, as the German sociologist Norbert Elias (1993, p. 32) states, "developed his individual possibilities of giving expression to feelings within the framework of the old [musical] canons within which he had grown up." His genius combines the novelty he expresses from the established canons. His music is accessible and attractive insofar as it resonates with the musical canons of the time while presenting new forms of expression and musical imagination.

The roots of knowledge

From the dialectical materialist perspective that I consider here, knowledge as disposition/potentiality is understood as a dynamic system whose roots are found in human cultural-historical activity.

Thus, a circle is not an *a priori* conceptual category; that is, a category that would exist *prior to* and *independently of* the experience that individuals have of the world. The proposition "the sun is round," for example, is not the product of a deduction of *a priori* categories of thought. It is quite the opposite: "People could see the sun as round only because they rounded clay with their hands" (Mikhailov, 1980, p. 199). If we see lines, rectangles, etc., around us, it is because with their hands, our ancestors gave shape

> to stone, sharpened its edges, gave it facets… The meaning of the words "border," "facet," "line" does not come from abstracting the general external features of things in the process of contemplation (Mikhailov, 1980, p. 199).

Conceived in this way, knowledge in general, and mathematical knowledge in particular, is not a psychological entity sunk into the individual's mind; neither is it a construction through which "a teleological [universal] reason running throughout all historicity announces itself," as Husserl claims in his *The Origins of Geometry* (Husserl, in Derrida, 1989, p. 180). Knowledge is a cultural-historical entity that must be understood as movement, as a "*process*" (Hegel, 2012, p. 262) imbricated in the life of individuals and

produced in that life, a "dialectical process that eternally separates and differentiates the identical in itself from the different, the subjective from the objective, the finite from the infinite, the soul from the body" (p. 261).

Knowledge as the bearer of its own contradictions

That knowledge is a dialectical process means, in particular, that it is constituted in and through its relations and properties with other things. Let us look at an example, that of the circle.

Circle is the name of that geometric knowledge or idea that is constituted through relations of commonality and differences with other forms. If, in our experience of the world, what appeared before us were only identical circles, we could not properly come to know such a geometric form. For this, we need *difference*: we need to differentiate it from that which the circle *is not*, such as a line or a triangle, for example.

In more general terms, knowledge is not a simple formal unit, as would be the genus-species, for example, but a *differential unity* of

> the positive and the negative, so that the positive is in such a way the identical relation to itself, that it *is not* the negative, and that the negative is in such a way the different for itself, that it *is not* the positive. (Hegel, 2012, p. 196; emphasis in the original)[1]

That is why knowledge, as I am outlining it here, is the bearer of its negation: it is the bearer of that which is not itself and which, not being itself, is a constitutive *part* of itself. It is precisely the inevitably negative nature of knowledge—its distinction from what is not it (other geometric forms, such as the line or the triangle, etc., in our example) that opens paths for the continuation of its process of complexification (transformation, generalization, etc.). That is why knowledge is *becoming*.

Knowledge as a unity of the abstract and the concrete

In the conception of knowledge presented here, another important element of its ontological constitution is that knowledge is both *abstract* and *concrete.*

Referring to the geometrical circle, Ilyenkov describes the situation as follows: the idea of a circle appears through "the conscious state of our body *identical in form with the thing [the circle] outside the body*. (1977, p. 69). We see here knowledge as an abstract and concrete entity at work. Its abstract nature is *unveiled* and *fixed* in its semiotic, material, and bodily *concreteness*, in the singularity of the circle produced, without therefore vanishing or disappearing from it.

> "This," Ilyenkov tells us, "can be represented quite clearly."

1. The translations in this article are free translations.

> When I describe a circle with my hand on a piece of paper (in real space), my body … comes into a state fully identical with the form of the circle outside my body, into a state of real *action* in the form of a circle. My body (my hand) really describes a circle, and the awareness of this state (i.e., of the form of my own action in the form of the thing) is [the circle]. (Ilyenkov, 1977, p. 69; emphasis in the original)

As we see, there are three elements in Ilyenkov's account: a) embodied activity; b) the circle as an ideal form, and c) consciousness as the link between a) and b). Notice that it is not a consciousness of the mere empirical action, but of the *form* of the action in the form of the thing. To remain within the consciousness of the action is to remain confined to empiricism—the first figure of consciousness in Hegel's (1977) account. Consciousness must move beyond the embodied act itself (without however leaving it behind, but rather bringing it along in a sublating sense of negativity), and generalize the pure singularity of it, which, as Vygotsky (1987) recognized, following Hegel, requires language; that is to say, a discursive social, cultural, and historical practice.

This is why the unity *abstract-concrete* can only be understood through three interrelated elements. First, the abstract-concrete unity must be understood as a *historical-cultural unity*, which is what Mikhailov's remarks mentioned above point to. Second, the body that describes the circle in Ilyenkov's example is not the *simpliciter* body (e.g., that of the acultural and ahistorical individual of other philosophies), but the body of the *social-historical individual.* Third, the ideal form of knowledge is not the result of the sensory impression with which objects mark the mind, but "is the form of a thing created by social-human labor, reproducing forms of the objective material world" (Ilyenkov, 2012, p. 191). Or to put it another way, it is "the form of labor realized in the substance of nature," (p. 191), labor that is embodied and realized in it.

Knowing

In the TO, the theoretical construct of *knowing* has a different meaning than the one usually attributed to it in other theories. Knowing refers to the unity of the abstract and the concrete mentioned in the previous section. Its purpose is functional: to help better understand learning as a process of encountering cultural knowledge. Let us see how.

It has been mentioned above that in the unity of the abstract and the concrete, knowledge is *revealed* and *fixed in* its semiotic, material, and corporal *concreteness*, in the singularity of the activity and its product: a circle, in our example. Let us imagine the following situation, which I extract from my fieldwork with kindergarten teachers and students. The teacher shows the figure of a circle on a poster placed on the blackboard. She groups the children in pairs and asks them to draw circles on a sheet of paper and then discuss differences and similarities in what they have drawn.

2. There is no exact term to translate knowing in French (and other languages, like Spanish and Portuguese). For lack of something better, in (Radford, 2020), knowing was translated as connaissance (see, e.g., Radford, 2020). One must bear in mind, though, that, as it will become clear shortly, knowing in the theory of objectification is not a subjective entity.

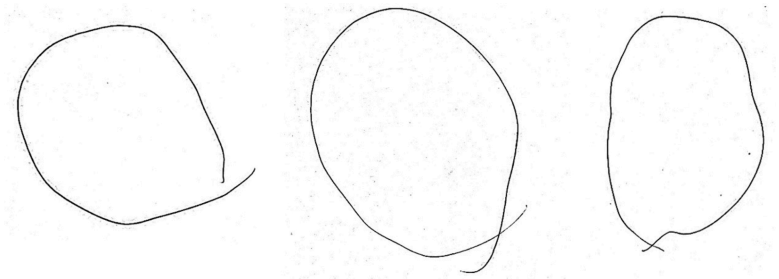

Figure 1. Three circles drawn by a 4-year-old girl.

Will knowledge coincide with the circles drawn? No. Those concrete figures drawn on paper, those signs, are just that: signs or representations of the circle as cultural knowledge. If those drawn circles are not knowledge, are they knowing? No. Knowing is a *process*: the semiotic, discursive, perceptual, tactile, kinesthetic process through which knowledge—in our example, the cultural idea "circle"—is manifested.

The form of being of the manifestation of knowledge is *concretion*. Following Hegel's terminology, this concretion is the abstract in a *developed form*; that is, in a sensible, concrete form of something abstract that, as such, was insinuated only as *potentiality* or *disposition*.

Knowing should not be understood as a "copy" or "reproduction" or "representation" of knowledge. The relationship between the abstract and the concrete, between knowledge and knowing, is not semiotic. It is dialectical. This means that knowing must rather be understood as knowledge in a new mode of existence: a mode of *real* existence in which, precisely thanks to its concreteness, differences become visible. It is only in this new mode of existence that the immanent contradictions of knowledge mentioned above can operate, so that, now, changes and transformations become possible and, from these, the discovery of new patterns of action and reflection. As Magee (2010) notes, in the field of the concrete—the field of knowing—individuals can discover and reflect on new things and create new potentialities.

We can resort to various linguistic terms to express the relationship between knowledge and knowing. For example, we can say that knowledge *manifests itself* in knowing, or is *concretized* or *embodied* or *materialized* in it. Language shows here its insufficiency to account for the phenomenon in question. Each of these terms refers us to a metaphor, which is both illuminating and inadequate—inadequate in that it tends to obscure the fact that the manifestation of knowledge is not a simple appearance of the abstract but a *dialectical unity* of the abstract and the concrete. The world in which we live, what Hegel calls *Wirklichkeit*, the *effective reality*, is a world that is both ideal and material at the same time (Vieillard-Baron, 2005). It is a world where a complex of historical-cultural ideas moves incessantly through the various human activities. In these activities, individuals produce things to satisfy their needs, which includes new ideas that prolong, refine or object to previous ideas. That is why the empirical world cannot be reduced to the tangible concrete itself. The circles produced by the children in our example are already impregnated with the ideal form, the cultural-historical idea of the circle and the activity of its concretization. Let us look for a moment at our surroundings. What

do we see? A desk, a chair, a bookcase, a window, a garden, a street, and so on. These things are concretions of cultural-historical ideas. In reality, they are more than that: they are concretions of ways of thinking and living in the world.

That is why I must better specify the conception of knowledge that I am outlining. Previously I proposed that knowledge could be conceived as a complex of ideas. We now see that it is more accurate to say that *knowledge is a dynamic system of historically and culturally constituted ways of thinking, acting, talking, and reflecting on the world.* These forms manifest themselves in sensible, material, discursive, and bodily processes through human activity, processes that I have called knowing.

Learning

The fact that cultural knowledge is continually manifesting itself in the ways we live and understand the world does not mean that knowledge is immediately accessible to each of its individuals. This is particularly true of what we might call "specific" knowledges, knowledges that respond to very precise activities of the culture, such as the financial knowledges developed by bankers and their slaves in Athens in the classical period (see Radford, 2021). Another example is that of contemporary scientific knowledge. As Christine Howe notes in her commentary on scientific knowledge, left to themselves, children

> are not going to [re]construct Newton's laws or Darwin's theory of evolution, nor, given the difficulties that adults are known to experience … are they going to master the full intricacies of hypothesis testing. (2009, p. 93)

In order to access this knowledge, it will be necessary to undergo a process of learning. In the TO, learning is conceived as an *encounter* with this cultural knowledge. This knowledge already exists in the culture, it is impregnated in the materiality of the world, but it is necessary to notice it, to see it, to understand it in its own logic.

Let us return to Ilyenkov's text. The Russian philosopher speaks of a state of "full identity" (2012, p. 69) between the action of the body and the ideal form of the circle. The situation described by Ilyenkov would seem incomprehensible. How can there be full identity between two such different things as a bodily action and an ideal form? Indeed, from the perspective of Aristotelian logic, these two elements are incommensurable. But this is precisely not the case in dialectical materialism, which relies here on an idea of Spinoza and another of Hegel. For Spinoza, corporeal material action and the idea of action are two sides of the same coin. In his *Ethics*, Spinoza considers the body as an extensive thing that expresses itself continuously through modes of extension. Spinoza says: "a mode of extension and the idea of that mode are one and the same thing, though expressed in two ways" (1989, p. 83). For example, the circle drawn by the hand—which is a mode of extension—and "the idea of a circle […] are one and the same thing displayed through different attributes" (Spinoza, 1989, p. 83).

Evidently, not just any mode of extension, not just any figure that draws the hand, is suitable. There must be *adequacy* between the two. It is this adequacy that Ilyenkov thematizes in Hegelian terms, speaking of "full identity" between idea and bodily action. This identity must be understood in terms of *differences.* As Magee (2010, p. 30) explains, "'A is A,' a thing is what it is. Hegel points out, however, that identity is a meaningless abstraction without the concept of difference." Ilyenkov quickly adds that it is not only identity but "consciousness" of this state of identity/difference (2012, p. 69). It is in this sense that Ilyenkov should be understood when he speaks of a *state of consciousness* of full identity between bodily idea and action.

For the mathematician, "the awareness of this identity" between action and ideality is self-evident, to the point that when drawing a circle in a demonstration process, it is not questioned. However, this is not the case with children in kindergarten classes.

Of course, these children have seen many circular shapes in their environment before—at home, in the street, etc. It is now a matter of seeing the circle *through new eyes*. It is about encountering or re-encountering it through a new experience that leads to a new awareness (*prise de conscience*) that Vygotsky (2019) thematizes through his distinction between everyday concepts and scientific concepts, a distinction similar to the one provided by Davydov (1990) in his difference between empirical thinking and theoretical thinking. There is a cultural way of thinking about the circle that does not readily show itself to consciousness.

For this cultural way of thinking about the circle to show itself, it will be necessary to create precise pedagogical conditions. There will be a need for an *activity* through which knowledge manifests itself as knowing. A way of thinking the world cannot, in fact, present itself; it always presents itself *mediated,* in its concreteness.

This means that learning—the encounter with cultural knowledge—passes through the concretions of knowledge—through knowing. Indeed, it is through knowing that knowledge can become an object of consciousness.

In the theory of objectification, the investigation of learning as encounter is done through the investigation of *processes of objectification*. These are the social processes of becoming aware of a cultural-historical form of thinking and action that reveals itself to consciousness through our full participation, mobilizing creativity and imagination: a bodily, sensuous, and artefactual semiotic activity from which ideas, understandings, and contradictions emerge as we thematize them within the limits and possibilities of the cultural-historical expressivity of semiotic systems, artefacts and the kinesthetic movement of our body (Radford, 2021). In fact, the term "objectification" present in the name of the theory refers precisely to that encounter, the encounter with something or someone that is not me. It is the encounter with that which objects, interrogates, and questions me; it is the encounter with the other, with alterity and otherness (Radford, 2023).

It should be noted that in the conceptualization of learning that I propose, consciousness is not conceived as a simple repository. Consciousness is not taken as already given in its

essentialities, but as *creative movement*. This is why learning cannot consist of pouring cultural meanings into the consciousness of a passive recipient learner. As the Brazilian educator Paulo Freire argues, "The importance of consciousness lies in the fact that, not being the creator of reality, neither is it, at the opposite pole, a pure reflection of reality" (Freire, 2014, p. 92).

When confronted with what objects it, consciousness moves in a dialectical movement in which it is *intertwined* with the world and with the ideas and meanings it encounters and produces from that world. To say that the movement of consciousness is a dialectical movement does not mean that the consciousness enters into a reciprocal play of effects with the world in order to reach a state of equilibrium. It means the creation of a new element, a new *unity* in which its elements now appear related to each other. Vygotsky gave water as an example. Water is not simply the sum of hydrogen and oxygen. It is a new *unity*. This is how consciousness should be understood. In learning, consciousness is *transformed* by its encounter with knowledge while at the same time opens up possibilities for its transformation. Instead of being simply a device for monitoring and tracking what is happening around the individual, consciousness is the *relationship* of the individual "with reality" (Vygotsky, 1997, p. 211). Consciousness can only be understood as the product of relations that are emergent and contingent, relations that are always operating through cultural-historical mediations that, rather than being simply given, "arise in the course of the establishment and development of society" (Leont'ev, 1978, p. 79). From this viewpoint, consciousness appears in concrete life, not as its origin, but as its result. There appear what can be termed, following Hegel (1977), new figures of consciousness, each of them imbricated in the world in different ways: Mozart as court musician and Mozart in search of his emancipation from the aristocracy in the context of the political and economic contradictions reflected in the bourgeois groups of his time, are, indeed, two figures of consciousness.

Activity

In the previous section, I followed a specific approach to define learning: I did so through processes of objectification; that is, the creative processes of becoming aware of the meanings of the cultural logic of knowledge. However, the processes of objectification do not constitute the unit of analysis to investigate learning.

It is important to note that any theory (educational, psychological, anthropological, etc.) that investigates learning resorts (implicitly or explicitly) to a *unit of analysis*. This unit of analysis serves to explain how learning occurs: it is the *explanans* of the *explanandum* (see Figure 2).

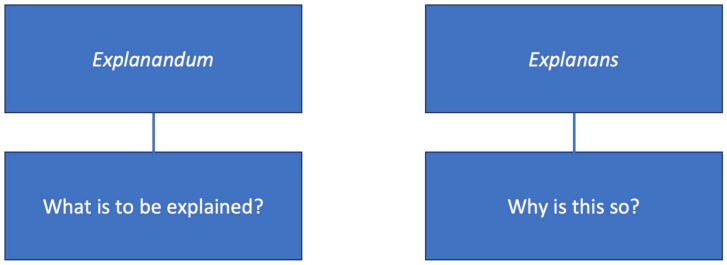

Figure 2. The unit of analysis (*explanans*) is used to explain the phenomenon under study (*explanandum*).

The unit of analysis integrates the theoretical principles of a theory, the type of research questions it poses, and the methodology (Radford, 2008). For example, in his genetic epistemology, to explain the cognitive development of the child, Piaget takes as the unit of analysis the actions that the child performs in front of a problem and interprets and explains them from structures of intelligence that he considers *inherent* to the human being. "From the beginning, intelligence is oriented towards reversibility, which increases in importance throughout development" (Piaget, 1965, p. 16). "Intelligence is *spontaneously oriented* toward the organization of *certain* operational structures" (Piaget, 1965, p. 31; emphasis added). Similarly, Piagetian constructivism takes the actions of the learner as the unit of analysis to study learning.

Dialectical materialism takes a different approach. In this perspective, the understanding and explanation of social phenomena—political, aesthetic, and economic, for example—are to be found in the *activities* of the individuals. And this is particularly true of psychological and educational phenomena, for, as Davydov explains, a person's thinking "is the functioning of historically developed forms of society's activity which have been *conferred* on him [sic]" (Davydov, 1990, p. 232). Therefore, for dialectical materialism, the sources of thought and development are not in the immanence of universal structures of intelligence, but in cultural-historical activities.

In the theory of objectification, following this idea of dialectical materialism, the unit of analysis to *explain* learning is *activity*: the teaching-learning activity in which the processes of objectification take place.

Activity does not have a single meaning, even within educational theories that draw on the work of Leont'ev (1978). For example, Rogalski (2013) distinguishes students' learning activity from teachers' practices. Davydov (1999) emphasizes theoretical knowledge and student activity as a process of appropriation of that knowledge, which leads him to talk about *learning activity*. From a similar perspective focused on theoretical thinking, Moura et al. (2010) propose the concept of *teaching orienting activity* (actividade orientadora de ensino). A detailed discussion of the differences and similarities of the concept of activity in these and other approaches is beyond the scope of this article. I will limit myself to mention, in general terms, some aspects of the concept of activity as used in the theory of objectification (for a more detailed discussion, see Radford, 2022a).

First, activity is not seen as purely *technical* or *instrumental*—that is, as a series of individual or group actions to achieve an objective. Second, activity is not seen as purely subjective, as when one says "the student's activity" or "the teacher's activity." In dialectical materialism, *activity is the process by which individuals endlessly constitute themselves as they engage daily in the historical and cultural world*. Therefore, what characterizes the individuals is not an inherent substantiality that they would possess of their own, but the forms of activity in which they shape their life (Fischbach, 2014). This is particularly true in educational activities, including those that take place in the mathematics classroom.

Hegel is recognized as one of the first philosophers who attributed a fundamental importance to the idea of activity, which in his work *Phenomenology of Spirit* (Hegel, 1977), is theorized under the concept of "work" (*travail*). Renault states:

> Hegel could be considered the founder of the first philosophically significant conception of work. He was certainly one of the first writers to grant it a fundamental importance, since work, for him, became one of the essential forms of relation to oneself and to the world, as well as a social activity in itself. (Renault, 2016, p. 469)

For Hegel, "activity is a process through which a new kind of self-consciousness can emerge that makes freedom possible" (Renault, 2016, p. 487). And this is so because activity, in the Hegelian sense, implies negation, for to act is to do something, and "to do [something] is [already] to deny what is" (Bourgeois, 2000, p. 30).

In the theory of objectification, to study learning, activity is taken as the unit of analysis. The processes of awareness (*les processus de prise de conscience*) is investigated as collective processes (synchronous or asynchronous) in which teachers and students collaborate *together* in the concretion of knowledge. In this approach, we break with the traditional dividing line between teachers and students. Although the teacher does not perform the same actions as the students, he or she does not act as an antagonist. We study how, through their *joint activity*, teachers and students seek to make sense of cultural knowledge in an encounter that is intended to be enriching in terms of human experience, cognitively, socially, and ethically. Therefore, the unit of analysis is not the students' activity on the one hand and the teacher's activity on the other, but a single activity: the *teaching-learning activity*.

In the example developed in (Radford, 2022b), a teaching-learning activity in a third grade elementary school class (8-9-year-old students) that was part of a 5-year longitudinal project is analyzed. The activity began with an introductory discussion in which the teacher sought to create the conditions for an encounter with algebraic knowledge related to the solution of equations. In this case, the teacher and the students discussed the equation $3 + x = 7$, presented through a story and represented in a semiotic way with cardboard cards and an envelope containing an unknown number of cards (see Figure 3).

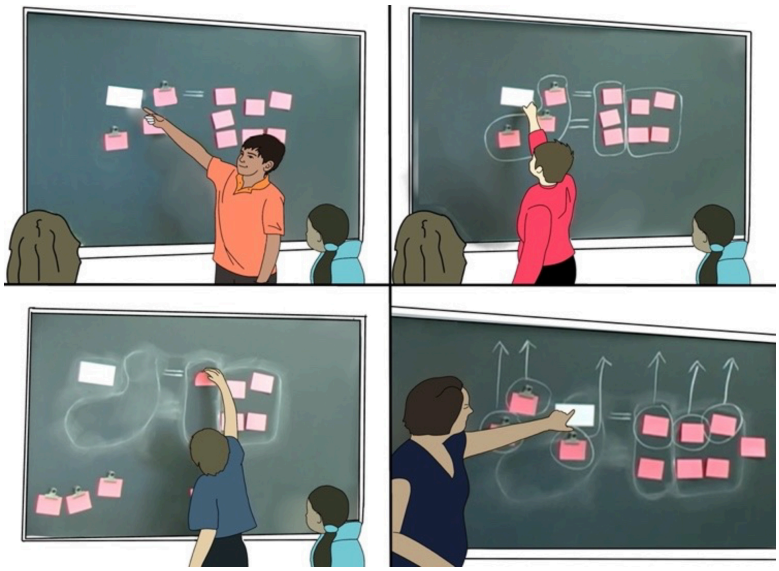

Figure 3. The joint activity of the teacher and the students in the concretion of knowledge.

At first, the students suggested various arithmetic procedures to solve the equation. Figure 3, left, above, shows Jase, who, through trial and error, determined that there were 4 cards in the envelope. Jase explained to the class, “4 (pointing to the envelope) plus 3 equals 7 (pointing to the 7 cards).” The teacher appreciated Jase’s idea and invited the class to come up with other ideas. Figure 3, above right, shows William, who used a different arithmetic procedure, the matching procedure. William circled a set of 3 cards on the left side of the equation and 3 cards on the right side. He drew a second equal sign to indicate that the 3 cards on the left side were equal to the 3 cards on the right side. “So,” he said, “this here (pointing to the remaining 4 cards) must be equal to this (pointing to the envelope; see the right side, top, of Figure 3). The envelope should contain 4 cards.”

As can be seen, in third grade, the procedure of isolating the unknown was not the students’ first choice. The teacher had to ask, referring to what they had learned in second grade, “What do we mean by *isolating*? If I tell you that I would like to isolate the envelope…” Cyr, one of the students replied, “Does it mean … like putting it alone?” When the teacher asked Cyr to explain his idea, Cyr walked over to the board (see Figure 3, below, left) and began removing one card at a time from each side of the equation. The procedure of isolating the unknown was shown with actions rather than articulated in words. The teacher rephrased Cyr’s actions (see Figure 3, bottom, right):

1. Teacher: Okay, wait… Let’s remove them one at a time, okay? We remove one, but that (*pointing to the equal sign*) says “equal”; that the number of cards on both sides is equal; so, if you remove one [card] from this side, what do you do?
2. Cyr: I remove another one from there (*removes a card from the other side of the equation*).
3. Teacher: You have to remove one from this side (*pointing to the right*).
4. Cyr: Then I would remove another one from here (*left side*) … another one from here (*right side)*. The last card from here (*left side*) and one last card from here (*right side*) and then there’s…
5. Teacher: (*addressing the class*) So what happens? Is the envelope isolated? Is the envelope alone?

6. Students: Yes!
7. Cyr: So, I would count how many cards there are … 4!
8. Teacher: That means the envelope (*points to the envelope*) is equal to (*points to the equal sign*) … how many cards are there [in the envelope]?
9. Mariana: 4!

In this excerpt from the teaching-learning activity, the teacher and Cyr worked *together* to bring the algebraic idea of isolating the unknown to life. Working with Cyr at the board and continually addressing the class, the teacher highlighted Cyr's actions through concrete objects, gestures, and language, striving to help students reach a more profound conceptual understanding of the ideas behind the algebraic procedure. Until that morning, for these students, the idea was potential—a cultural disposition. The idea manifested itself through joint activity, in a process of objectification: a discursive, semiotic, bodily, and material process that gradually acquired meaning. In this concretization, knowledge *manifested* itself through differences and contradictions, it emerged together with what it was not: it was not trial and error, nor was it a comparison of terms. It was something different…

The class was then divided into small groups of three students. The teacher encouraged students to participate in the activity and collaborate with others to forge together an ethic of inclusion, commitment, and solidarity in solving increasingly complex problems. From that ethical and conceptual work gradually emerged an awareness (*prise de conscience*) of the operations required in the procedure to isolate the unknown when simplifying linear equations with positive integer coefficients.

The procedure for isolating the unknown was a key aspect of the systematization of algebra carried out by Arab mathematicians in the eighth and ninth centuries (such as Al-Khwārizmī and others; see Oaks and Alkhateeb, 2007). It involved operations with known and unknown quantities to simplify equations. Arab mathematicians called these operations *al-gabr* and *al-muqābala*, and our modern term algebra is derived from the former. In this third grade class, the procedure for isolating the unknown was not only presented as simultaneously abstract and concrete, but also as simultaneously old and new: old in its historicity and new in its current presentation, and in the new possibilities it offered for further advancement, a bit like Mozart's music.

Concept

Both knowledge and its materialization—that is, knowing—are, as mentioned above, cultural-historical entities. However, the *understanding* of knowledge derived from collective activity differs from one student to another. To address these differences, the construct of *concept* is used in the theory of objectification.

Usually, a concept is considered to be a type of mental entity. However, this does not apply here. Like other notions within the theory of objectification, the construct of concept must be understood dialectically, that is, in relation to its origin and transformation from the activity from which it arises.

Following this line of thought, it can be noticed that, as knowledge is put into action through collective activity, it materializes into something tangible, what I call knowing. Throughout this process of materialization, knowledge, through knowing, is *refracted* in the consciousness of the students. This refraction is always different and varies from one student to another. Therefore, at the end of the teaching-learning activity discussed above, William's understanding of the operations involved in the process of isolating the unknown may be different from that of Jase. A concept is just that: *the subjective refraction of knowledge into consciousness through the mediation of knowing*. A concept enables us to carry out actions and think in certain ways. It is something that, like language, we do not possess, but come to enjoy.

The idea of *refraction* in the definition of the concept emphasizes the fact that the concept is an entity that is at the same time subjective and objective, material and ideal. It is not simply a *reflection*, such as the image reflected by a mirror. The term refraction, as used here, implies *hybridization*, the creation of a new entity. A concept is just that: a hybrid, subjective-objective entity, something that offers real possibilities for action: like a new "organ" that allows the individual to interact in a cultural context with others. A concept *connects* the learner with culture and history and, at the same time, transforms him/her into a cultural-historical subject (Radford, 2021).

Knowledge, knowing, and concept are the same entity in three different modes of existence. They are the same in their differences and different in their similarities. Thus, knowledge is a formless form that can only be attained through knowing. Knowing manifests knowledge in an always partial way, affirming and denying it in its manifestation. It affirms it in its positivity, in what it shows: the *actions, ideas, phrases, and gestures* that appear in the dialogue between the teacher and her class about how to solve algebraically a concrete equation, such as the equation $3 + x = 7$; and it denies it by leaving out other aspects. Knowing can never present knowledge in its tangible (evolving) totality. Likewise, a concept is always *precarious*, for it will always be partial in what it gathers from knowledge through knowing. This is why knowing and concept are *deficit* vis-à-vis knowledge. But they are at the same time an *excess*, for the growth and transformation of knowledge can only take place through knowing and concept, since it is only in social life and in effective reality, in *Wirklichkeit*, that knowledge can be formed and constituted.

By way of summary

The aim of this article has been to present some key notions that serve to understand learning from a cultural-historical perspective. I started by recalling that contemporary socio-cultural currents in education, psychology, anthropology, and other related fields affirm that knowledge is intrinsically *cultural*. However, *theoretical* explanations of the cultural nature of knowledge have been one of the main challenges for these approaches, especially in education, due in part to the Western tradition that has defended a universalist or empiricist conception of knowledge.

This article has deliberately followed a theoretical line. Readers interested in a reflection on the didactic implications may see Radford et al. (2023); for concrete examples of investigations of learning, see Radford (2022 b). Adopting a dialectical materialist perspective, the first part of the article presented a conception of knowledge as a system of historically and culturally constituted forms of thinking, action, and reflection. Ontogenetically speaking, knowledge appears as disposition or potentiality. Instead of being platonic forms or abstractions of an empirical subject, its roots are in human cultural-historical activity.

Dialectically conceived, knowledge is not a simple formal unity, as would be the genus-species, but a *differential unity* that carries with it the negations inherent in the activity or work of social individuals. Knowledge manifests itself in that activity in a tangible way through the body, artefacts, signs, language, etc., which makes it a unity of the abstract and the concrete, of the old and the new, since history is understood here "not as determinism but as possibility" (Freire, 2014, p. 82).

It is manifested knowledge that I call knowing. Learning is the process of becoming aware of knowledge as presented by knowing. I have called these processes of becoming aware processes of objectification, emphasizing that they need to be understood dialectically, that is, not as passive processes, but as processes of active individuals anchored in creativity and imagination, processes in which personal meanings and cultural meanings converge, diverge, and intertwine.

The processes of objectification occur in human activity, specifically in the teaching-learning activity and where the hyphen "–" signifies union, that is, co-occurrence. In these activities, the gap that separates the teacher and the students fades away to include them both in the pursuit of a common goal, the knowledge to be learned. In this sense, teaching-learning activity is a collective activity in which teachers and students, suffering and enjoying, work together towards the realization of a *common work*: the concretization of knowledge and the revelation of its cultural logics.

In the second part, the focus was on the explanation of how learning occurs. I argue that this explanation is found in teaching-learning activity, which becomes the unit of analysis. This unit of analysis shows us that alienating activities will inevitably lead to alienating learning, while enriching activities (in the sense of providing opportunities for deep conceptual understanding and supportive and inclusive social experiences) will lead to rich learning (Radford, 2012).

This discussion led us to consider the notion of concept. A concept is not a purely subjective entity; it is, on the contrary, a dialectical union of the objective and the subjective, the subjective refraction of knowledge in consciousness through the mediation of knowledge.

Knowledge, knowing, and concept are always present in each of our actions, in each of our thoughts, from the very beginning of our day. They are the same entity in three different modes of existence. They are present both in our moments of interaction with

others and in our moments of solitary reflection. Every teaching-learning activity draws on the regions of knowledge with which we are familiar. And it is from this familiar knowledge that the dialectics that leads us to encounter new knowledges is woven.

References

Balibar, É. (2002). *Politics and the other scene*. Verso.

Barbin, E. (2009). *De grands défis mathématiques d ' Euclide à Condorcet.* Vuibert.

Bkouche, R. (1997). Épistémologie, histoire et enseignement des mathématiques. *For the Learning of Mathematics, 17*(1), 34–42.

Bourgeois, B. (2000). *Le vocabulaire de Hegel*. Ellipses.

Davydov, V. (1990). Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula. *Soviet studies in mathematics education*, vol. 2. NCTM.

Davydov, V. (1999). What is real learning activity? In M. Hedegaard and J. Lompscher (Eds.), *Learning and development* (pp. 123–138). Aarhus University Press.

de Freitas, E., Sinclair, N. and Coles, A. (2017). *What is a mathematical concept?* Cambridge University Press.

de Moura, M., Araújo, E., Moretti, V., Panossian, M. L. and Ribeiro, F. (2010). Actividade orientadora de ensino: unidade entre ensino e aprendizagem. *Rev. Diálogo Educ., Curitiba, 29*(10), 205–229.

Derrida, J. (1989). *Edmund Husserl ' s origin of geometry: An introduction*. University of Nebraska Press.

Elias, N. (1993). *Mozart. Portrait of a genius*. University of California Press.

Fischbach, F. (2014). *La production des hommes. Marx avec Spinoza*. Vrin.

Freire, P. (2014). *Pedagogy of hope*. Bloomsbury.

Fried, M. (2009). Similarity and equality in Greek mathematics: Semiotics, history of mathematics, and mathematics education. *For the Learning of Mathematics, 29*(1), 2–7.

Hegel, G. (1969). *Science of logic*. Humanity Books.

Hegel, G. (1977). *Hegel's phenomenology of spirit*. Oxford University Press.

Hegel, G. (2012). *Encyclopédie des sciences philosophiques en abrégé*. Vrin.

Howe, C. (2009). The role of the teacher in the transformation of knowledge in classroom interaction. In M. Schwarz, T. Dreyfus and R. Hershkowitz (eds.), *Transformation of knowledge through classroom interaction* (pp. 93–104). Routledge.

Husserl, E. (1972). *Philosophie de l ' arithmétique* (J. English, Trans.). Presses universitaires de France.

Ilyenkov, E. (1977). *Dialectical logic*. Progress Publishers.

Ilyenkov, E. (2012). Dialectics of the ideal. *Historical materialism, 20*(2), 149–193.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Prentice-Hall.

Levant, A. (2016). Smart matter and the thinking body. *Stasis, 5*(2), 248–264.

Lunney Borden, L. (2013). What's the word for...? Is there a word for...? How understanding Mi'kmaw language can help support Mi'kmaw learners in mathematics. *Mathematics Education Research Journal, 25,* 5–22.

Magee, G. (2010). *The Hegel dictionary*. Continuum.

Mikhailov, F. T. (1980). *The riddle of the self.* Progress Publishers.

Oaks, J. and Alkhateeb, H. (2007). Simplifying equations in Arabic algebra. *Historia Mathematica, 34*(1), 45–61. DOI: 10.1016/j.hm.2006.02.006 .

Otte, M. (2003). Does mathematics have objects? In what sense? *Synthese, 134*(1–2), 181–216.

Parra, A. (2018). *Curupira ' s walk: Prowling ethnomathematics theory through decoloniality* [PhD Dissertation, Aalborg University].

Piaget, J. (1965). Mathematical structures and the operational structures of intelligence. In C. Gattegno, J. Piaget, E. Beth, J. Dieudonné, A. Linchnerowicz and G. Choquet (eds.), *L ' enseignement des mathématiques. New perspectives.* Delachaux and Niestlé.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM—The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics, 80*(1), 101–118.

Radford, L. (2020). Le concept de travail conjoint dans la théorie de l'objectivation [The concept of joint labor in the theory of objectification]. In M. Flores González, A. Kuzniak, A. Nechache and L. Vivier (eds.), *Cahiers du laboratoire de didactique André Revuz n° 21* (pp. 19-41). IREM de Paris.

Radford, L. (2021). *The theory of objectification. A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill/Sense. DOI: 10.1163/9789004459663 .

Radford, L. (2022a). Activité, apprenant(s), apprentissage. *Revue québécoise de didactique des mathématiques*, Numéro thématique 1, 134–157. https://rqdm.recherche.usherbrooke.ca/ojs/ojs-3.1.1-4/index.php/rqdm/article/view/70 (accessed April 5, 2024).

Radford, L. (2022b). Introducing equations in early algebra. *ZDM - Mathematics Education, 54,* 1151–1167. DOI: 10.1007/s11858-022-01422-x .

Radford, L. (2023). Politics, knowledge and ethics: the need to rethink the teaching-learning of mathematics. *EMP—Educação Matemática Pesquisa, 25*(2), 45–68.

Radford, L., Miranda, I. and Vergel, R. (2023). Savoir mathématique et action didactique dans la théorie de l'objectivation. *CEMeR - Caminhos da Educação Matemática em Revista*. 13(4), 75 – 93. https://periodicos.ifs.edu.br/periodicos/caminhos_da_educacao_matematica/issue/view/97

Renault, E. (2016). Hegel et le paradigme du travail. *Revue internationale de philosophie, 4*(278), 469–490.

Rogalski, J. (2013). Theory of activity and developmental frameworks for an analysis of teachers' practices and students' learning. In F. Vandebrouck (ed.), *Mathematics Classrooms: Students ' activities and teachers ' practices* (pp. 3–22). Sense Publishers.

Rosa, M., Shirley, L., Gavarrete, M. and Alangui, W. (2017). *Ethnomathematics and its diverse approaches for mathematics education*. Springer.

Spinoza, B. (1989). *Ethics including the improvement of the understanding*. Prometheus.

Vieillard-Baron, J.-L. (2005). La « Wirklichkeit » ou réalité effective dans les « Principes de la philosophie du droit » de Hegel. *Revue Philosophique de Louvain, tome 103*(3), 347–363.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Vygotsky, L. (1987). *Collected works (Vol. 1)*. Plenum.

Vygotsky, L. (1997). *Collected Works (Vol. 3). Problems of the theory and history of psychology*. Plenum.

Vygotski, L. (2019). *Pensée et langage*. La Dispute.

This article was written as part of a research program funded by the Social Sciences and Humanities Research Council of Canada / Le conseil de recherches en sciences humaines du Canada (SSHRC/CRSH).